Librations of a Parametrically Resonant Dual-Spin Satellite with an Energy Damper

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The damped librations of a parametrically resonant dual-spin satellite are treated analytically. The system consists of a gravity-oriented main body, a free, undriven rotor, and a damper mass connected to the main body. Two methods are devised: 1) a canonical transformation method for the analysis of the attitude dynamics of the satellite connected to a moving part, and 2) a canonical transformation method for solving the damped parametrically resonant librations. The damped librations in the neighborhood of the parametric resonance are solved analytically using the canonical transformation and the method of averaging. These librations are compared with the case of no parametric resonance and the configuration parameters of the satellite which make the attenuation faster are investigated. The analysis also provides a technique for obtaining the conditions for the pervasive (complete) damping of linearized motion of the present system with semidefinite damping.

Introduction

MUCH attention has been given to the librations of nonlinearly resonant rigid satellites. Kane[1] first noted the effects of nonlinear resonance on the attitude stability of gravity-oriented satellites. He has shown that infinitesimal roll-yaw motion can be increased by the finite pitch motion for a certain "critical" configuration of the rigid satellites, using Floquet theory in conjunction with digital computation. Employing the Canonical transformation method and the averaging method, Breakwell and Pringle[2] have given an analytical explanation for the effect of nonlinear resonance on the atttitude motion of rigid satellites and have shown that parametric resonance causes a large internal kinetic energy exchange phenomenon and leads to the unstable attitude motion of rigid satellites.

If energy dissipation is introduced into the satellites, this parametric resonance is expected to affect the attitude motion in a somewhat different manner. Using the averaging method and numerical computations, Pringle[3] has exploited the nonlinear resonance in damping librations of a particular satellite (where a pair of point masses are connected by a dissipative spring) and has shown that the damping in the roll mode is markedly improved by another mode which resonates parametrically with the roll mode and thus exchanges modal energy with it. Likins and Wrout[4] have suggested that the parametric resonance be exploited as an internal kinetic energy exchange mechanism to accelerate energy dissipation and consequently to attenuate satellite librations. As a preliminary step in their analysis, they investigated the bounds of librations for parametrically resonant rigid satellites by portraying these bounds numerically.

The object of the present paper is to investigate analytically the attenuation of librations of parametrically resonant satellites having finite mass and connected to an energy damper.

In the present analysis, the main vehicle (main body and rotor) is not assumed to be a point mass and the system contains a damper mass. Therefore the formulation of the canonical transformation method employed in Ref. 3 for the case of a point mass is not directly applicable and that employed in Ref. 2 for the case of a rigid satellite becomes very complex to apply because the deflection of the damper mass is coupled with the satellite attitude deviation. In order to avoid this difficulty of application of the canonical transformation method, a formulation is presented to describe the damper deflection with respect to a coordinate system with axes parallel to the orbital frame. The normal modes in the linear motion are then derived using the canonical transformation method.

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The parametrically resonant motion is treated employing the averaging method and is described by two modes in which a large nonlinear energy exchange occurs. These two parametrically resonant modes can be described by two action variables both varying rapidly with respect to time. Introducing a canonical transformation, these two action variables are transformed into two new action variables. The first is a rapidly time-varying and the second is a constant for the associated conservative system. This transformation is used to solve the parametrically resonant motion of the conservative system by means of elliptic integrals. Employing these solutions of the conservative system as generating solutions, the method of averaging is applied to the nonconservative case to solve for the action variable which was constant for the conservative system. This solution can be used to estimate the decrease in the Hamiltonian of the nonconservative system.

Accordingly, an approximate solution of the Hamiltonian is obtained analytically for the present nonconservative system. To clarify the features of the parametric resonance in damping librations, the damping behavior of the satellite attitude motion is shown for the case of parametric resonance and that of nonparametric resonance. Finally, the ap-

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proximate solution is compared with the solution obtained by numerically integrating the canonical equations for the variations of generalized coordinates and moments.

System Description

The dual-spin satellite system consists of a rigid body, a rigid rotor whose spin axis is one of the principal axes of inertia, $\hat{a}_3$, of the main body, and a rigid two-degree-of-freedom energy damper mass connected to the main body (Fig. 1). The mass center, G, of the satellite moves on a circular orbit under the purely gravitational force field of the center of attraction P. The constant orbit angular rate n is assumed to be $n=1$ and the unit of time is $\frac{1}{2}\pi$ orbit period. The orbit frame F^o $(G-\hat{x}_1, \hat{x}_2, \hat{x}_3)$ has its origin at G; the $\hat{x}_1$ axis is taken along the radius vector from P to G, the $\hat{x}_2$ axis is along the orbit velocity vector, and the $\hat{x}_3$ axis is along the orbital angular velocity vector. The combination of the main body and the rotor is referred to as the main vehicle. The frame F^v $(C-\hat{a}_1, \hat{a}_2, \hat{a}_3)$ which is fixed to the main vehicle has its origin at the mass center C of the main vehicle and the unit axes $\hat{a}_1$, $\hat{a}_2$ and $\hat{a}_3$ are fixed to the main body. It is assumed that these axes are the principal axes of the main vehicle and the corresponding moments of inertia are I_1, I_2 and I_3. Next, if the frame F^o is shifted to the orbit frame F^o_v $(C-\hat{y}_1, \hat{y}_2, \hat{y}_3)$ which has the origin at C and each unit axis $\hat{y}_j$ is parallel to $\hat{x}_j, J=1,2,3$, the frame F^o_v is related to the frame F^v by the three successive rotations $\theta_1 \rightarrow \theta_2 \rightarrow \theta_3$ as shown in Fig. 1. Then the vector r from C to the damper mass can be expressed by the following two equations

$$r=(r_1+\xi_1)\hat{a}_1+\xi_2\hat{a}_2+\xi_3\hat{a}_3 \tag{1}$$

and

$$r=\eta_1\hat{y}_1+\eta_2\hat{y}_2+\eta_3\hat{y}_3 \tag{2}$$

where r_1 is constant and ξ_1, which is assumed to be zero, is a fictitious coordinate.

The vectors $\xi=(r_1+\xi_1, \xi_2, \xi_3)^T$ and $\eta=(\eta_1, \eta_2, \eta_3)^T$ are related by

It is assumed that the elastic potential energy Ve and Rayleigh's dissipation function F can be expressed as follows

$$Ve=\tfrac{1}{2}\sum_{j=2}^{3} k_j\xi_j^2 \tag{4}$$

$$F=\tfrac{1}{2}\sum_{j=2}^{3} c_j\dot{\xi}_j^2 \tag{5}$$

where dot denotes differentiation with respect to normalized time (i.e., true anomaly) and k_j and c_j are spring constants and the damping coefficients.

Assuming that the torque applied to the rotor is zero, the equilibrium condition for the rotor is given by

$$\omega_3^M+\omega^s \triangleq r+1=\text{constant} \tag{6}$$

where ω_j^M $(j=1,2,3)$ are the components of the angular velocity of the main body with respect to inertial space in body axes $\hat{a}_j$ $(j=1,2,3)$, and ω^s is the component of angular velocity of the rotor about $\hat{a}_3$ axis with respect to the main body.

Hamiltonian Function of the System

The kinetic energy T and the gravitational potential energy V_g for the present satellite system are given by the following equations:[6]

$$T=\tfrac{1}{2}\left(\sum_{j=1}^{3} I_j\omega_j^{M2}+2I_3^R\omega_3^M\omega^s+I_3^R\omega^{s2}\right)+\frac{m}{2}\left[\sum_{j=1}^{3}\dot{\eta}_j^2+2(\eta_1\dot{\eta}_2-\dot{\eta}_1\eta_2)+\eta_1^2+\eta_2^2\right] \tag{7}$$

$$V_g=\frac{3}{2}\left[(I_1-I_2)\cos^2\theta_2\cdot\cos^2\theta_3+(I_3-I_2)\sin^2\theta_2\right]+\frac{m}{2}(-2\eta_1^2+\eta_2^2+\eta_3^2) \tag{8}$$

$$\begin{bmatrix} r_1+\xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}=\begin{bmatrix} \cos\theta_2\cdot\cos\theta_3 & \cos\theta_1\cdot\sin\theta_3+\sin\theta_1\cdot\sin\theta_2\cdot\cos\theta_3 & \sin\theta_1\cdot\sin\theta_3-\cos\theta_1\cdot\sin\theta_2\cdot\cos\theta_3 \\ -\cos\theta_2\cdot\sin\theta_3 & \cos\theta_1\cdot\cos\theta_3-\sin\theta_1\cdot\sin\theta_2\cdot\sin\theta_3 & \sin\theta_1\cdot\cos\theta_3+\cos\theta_1\cdot\sin\theta_2\cdot\sin\theta_3 \\ \sin\theta_2 & -\sin\theta_1\cdot\cos\theta_2 & \cos\theta_1\cdot\cos\theta_2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \tag{3}$$

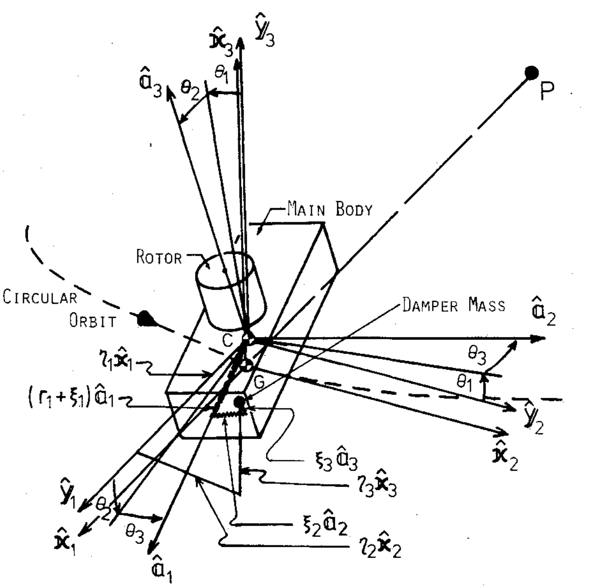

Fig. 1 A dual-spin satellite connected with a damper mass.

where

$$\omega_1^M=\dot{\theta}_1\cos\theta_2\cdot\cos\theta_3+\dot{\theta}_2\sin\theta_3+\sin\theta_1\cdot\sin\theta_3-\cos\theta_1\cdot\sin\theta_2\cdot\cos\theta_3$$

$$\omega_2^M=-\dot{\theta}_1\cos\theta_2\cdot\sin\theta_3+\dot{\theta}_2\cos\theta_3+\sin\theta_1\cdot\cos\theta_3+\cos\theta_1\cdot\sin\theta_2\cdot\sin\theta_3$$

$$\omega_3^M=\dot{\theta}_1\sin\theta_2+\dot{\theta}_3+\cos\theta_1\cdot\cos\theta_2$$

and I_3^R is the moment of the rotor about $\hat{a}$ axis, $m \triangleq M^D M^V/(M^D+rm^V)$ is the reduced mass, M^V is ths mass of the main vehicle and M^D is the mass of the damper.

The Routhian function R is given by

$$R\triangleq L-(\partial L/\partial\omega^s)\omega^s=T-Vg-Ve-I_3^R\omega^s(r+1) \tag{9}$$

where $L\triangleq T-Vg-Ve$ is the Lagrangian. The generalized momenta $'p_i$, $'p_2,\ldots,'p_6$ conjugate respectively to θ_1, θ_2, θ_3,

η_1, η_2, η_3 are defined by

$$'p_1 = \partial R/\partial\dot{\theta}_1 = I_1\omega_1^M\cos\theta_2\cdot\cos\theta_3 - I_2\omega_2^M\cos\theta_2\cdot\sin\theta_3 + [I_3\omega_3^M + I_3^R(r+1)]\sin\theta_2 \tag{10a}$$

$$'p_2 = \partial R/\partial\dot{\theta}_2 = I_1\omega_1^M\sin\theta_3 + I_2\omega_2^M\cos\theta_2 \tag{10b}$$

$$'p_3 = \partial R/\partial\dot{\theta}_3 - (I_3 - I_3^R)| = (I_3 - I_3^R)(\omega_3^M - 1)\,[\triangleq I_3^M(\omega_3^M - 1)] \tag{10c}$$

$$'p_4 = \partial R/\partial\dot{\eta}_1 = m(\dot{\eta}_1 - \eta_2) \tag{10d}$$

$$'p_5 = \partial R/\partial\dot{\eta}_2 - mr_1 = m(\dot{\eta}_2 + \eta_1 - r_1) \tag{10e}$$

$$'p_6 = \partial R/\partial\dot{\eta}_3 = m\dot{\eta}_3 \tag{10f}$$

Since the constraint, Eq. (3), is holonomic, the present motion can be investigated by setting the fictitious coordinate ξ_1 to zero, and by expressing η_1 (and $\dot{\eta}_1$, $'p_4$) in terms of other coordinates by Eq. (3) with $\xi_1 = 0$.

Introducing $'p_1,\ldots,'p_6$ and the above-stated holonomic constraint and collecting the terms of equal orders in the generalized coordinates and momenta, the Hamiltonian H of the system is obtained as follows:

$$H = H_2 + H_3 + \ldots \tag{11}$$

where

$$H_2 = \frac{1}{2I_1}\,'p_1^2 + \frac{1}{2I_2}\,'p_2^2 + \frac{1}{2I_3^M}\,'p_3^2 + \frac{1}{2m}\,'p_5^2 + \frac{1}{2m}\,'p_6^2 - \frac{I_3 + I_3^R r - I_1}{I_1}\,'p_1\theta_2 - {}'p_2\theta_1 + 3r_1\eta_2\theta_3 - 3r_1\eta_3\theta_2$$
$$- k_2r_1\eta_2\theta_3 + k_3r_1\theta_2\eta_3 + \frac{1}{2}\left[4I_1 - 4I_3 + I_3^R r + \frac{(I_3 + I_3^R r - I_1)^2}{I_1} - 3mr_1^2 + k_3r_1^2\right]\theta_2^2 + \tfrac{1}{2}(I_3 + I_3^R r)\theta_1^2$$
$$+ \tfrac{1}{2}(-3I_1 - 3I_2 - 3mr_1^2 + k_2r_1^2)\theta_3^2 + \tfrac{1}{2}k_2\eta_2^2 + \tfrac{1}{2}(m + k_3)\eta_3^2$$

$$H_3 = \left(\frac{1}{I_1} - \frac{1}{I_2}\right)'p_1\,'p_2\theta_3 - \frac{1}{I_1}\,'p_1\,'p_3\theta_2 + \left(\frac{1}{I_2} - \frac{1}{I_2}\right)(I_3 + I_3^R r)\,'p_2\theta_2\theta_3$$
$$+ \frac{1}{2}\,'p_3\theta_1^2 + \left(\frac{I_3 + I_3^R r}{I_1} - \frac{1}{2}\right)'p_3\theta_2^2$$
$$+ {}'p_5(\eta_2\theta_3 - \eta_3\theta_2) - \tfrac{1}{2}r_1\,'p_5(\theta_2^2 + \theta_3^2) - k_2r_1\eta_3\theta_1\theta_3$$
$$- k_3r_1\eta_2\theta_1\theta_2 + (k_2 - k_3)\eta_2\eta_3\theta_1$$

and the terms higher than the third order are neglected.

Linear Analysis and the Normal Modes

The Hamiltonian, H_2, consisting of the second order terms in the generalized coordinates and momenta, can be transformed into the following form

$$H_2 = \frac{1}{2}\sum_{j=1}^{5}(p_j^2 + \omega_j^2 q_j^2) \tag{12}$$

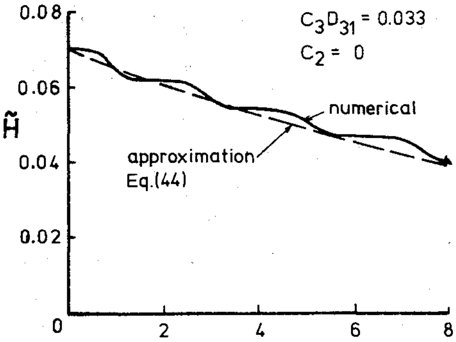

Fig. 2a Comparison between analytically and numerically predicted behaviors of $\tilde{H}$: plots against time in orbits.

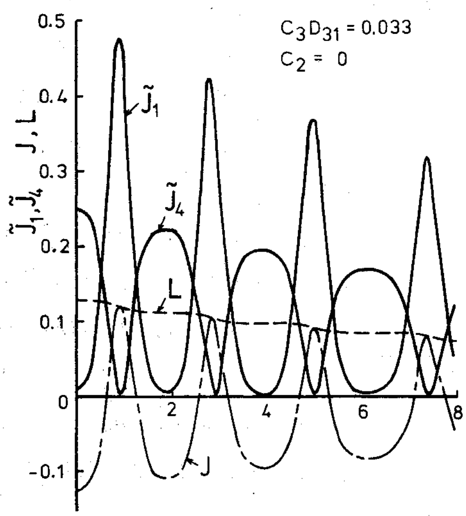

Fig. 2b Plots of $\tilde{J}_1$, $\tilde{J}_4$, J and L against time in orbits.

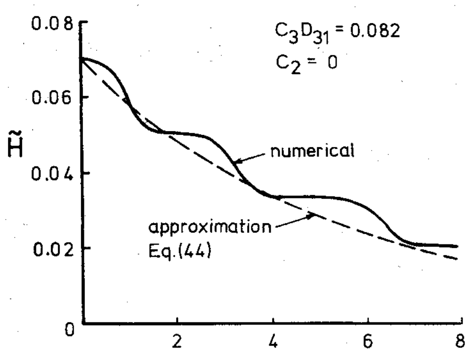

Fig. 3a Comparison between analytically and numerically predicted behaviors of $\tilde{H}$: plots against time in orbits.

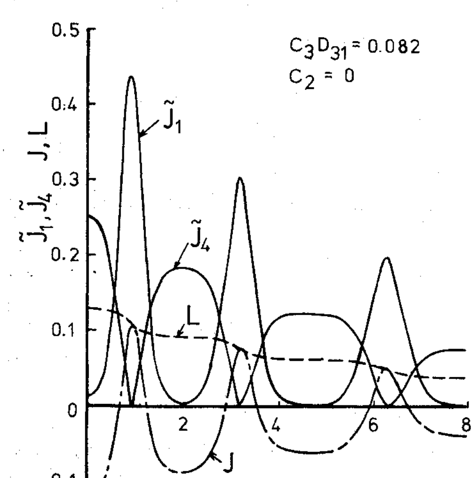

Fig. 3b Plots of $\tilde{J}_1$, $\tilde{J}_4$, J and L against time in orbits.

In the above expression, p_j, q_j and ω_j are calculated from the canonical transformations[7] given by

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \eta_3 \\ 'p_1 \\ 'p_2 \\ 'p_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & (2\rho_1/i\omega_1)C_{11} & (2\rho_2/i\omega_2)C_{21} & (2\rho_3/i\omega_3)C_{31} \\ 2\rho_1 C_{12} & 2\rho_2 C_{22} & 2\rho_3 C_{32} & 0 & 0 & 0 \\ 2\rho_1 C_{13} & 2\rho_2 C_{23} & 2\rho_3 C_{33} & 0 & 0 & 0 \\ 2\rho_1 C_{14} & 2\rho_2 C_{24} & 2\rho_3 C_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & (2\rho_1/i\omega_1)C_{15} & (2\rho_2/i\omega_2)C_{25} & (2\rho_3/i\omega_3)C_{35} \\ 0 & 0 & 0 & (2\rho_1/i\omega_1)C_{16} & (2\rho_2/i\omega_2)C_{26} & (2\rho_3/i\omega_3)C_{36} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} \theta_3 \\ \eta_2 \\ 'p_3 \\ 'p_5 \end{bmatrix} = \begin{bmatrix} \rho_4 d_{41}/(I_3^M)^{1/2} & \rho_5 d_{51}/(I_3^M)^{1/2} & 0 & 0 \\ \rho_4 d_{42}/\sqrt{m} & \rho_5 d_{52}/\sqrt{m} & 0 & 0 \\ 0 & 0 & \rho_4 d_{41}(I_3^M)^{1/2} & \rho_5 d_{51}(I_3^M)^{1/2} \\ 0 & 0 & \rho_4 d_{42}\sqrt{m} & \rho_5 d_{52}\sqrt{m} \end{bmatrix} \begin{bmatrix} q_4 \\ q_5 \\ p_4 \\ p_5 \end{bmatrix} \tag{14}$$

In Eq. (13), for $j=1, 2, 3$,

$$\begin{bmatrix} C_{j1} \\ C_{j2} \\ C_{j3} \\ C_{j4} \\ C_{j5} \\ C_{j6} \end{bmatrix} = \frac{1}{I_1 I_2 m} \begin{bmatrix} i\omega_j (I_1 + I_2 - I_3 - I_3^R r)(m\omega_j^2 - m - k_3) \\ -(I_1\omega_j^2 + I_2 - I_3 - I_3^R r)(m\omega_j^2 - m - k_3) \\ -(I_1\omega_j^2 + I_2 - I_3 - I_3^R r)(k_3 - 3m) r_1 \\ [I_1 I_2 \omega_j^2 - (I_1 - I_3 - I_3^R r)(I_2 - I_3 - I_3^R r)](m\omega_j^2 - m - k_3) \\ -i\omega_j I_1 I_2 (\omega_j^2 - 1)(m\omega_j^2 - m - k_3) \\ -i\omega_j (I_1\omega_j^2 + I_2 - I_3 - I_3^R r)(k_3 - 3m) m r_1 \end{bmatrix}$$

$\pm\omega_j$ ($j=1, 2, 3$) are the roots κ of the following equation

$$\begin{vmatrix} I_1\kappa^2 - I_2 + I_3 + I_3^R r & (I_1 + I_2 - I_3 - I_3^R r)\kappa & 0 \\ -(I_1 + I_2 - I_3 - I_3^R r)\kappa & I_2\kappa^2 + 4I_3 - 4I_1 + I_3^R r + (k_3 - 3m) r_1^2 & (k_3 - 3m) r_1 \\ 0 & (k_3 - 3m) r_1 & m\kappa^2 + m + k_3 \end{vmatrix} = 0 \tag{15}$$

ρ_j ($j=1,2,3$) are determined by

$$\frac{4\rho_j^2}{i\omega_j}(-C_{j1}C_{j4} + C_{j2}C_{j5} + C_{j3}C_{j6}) = 1$$

In Eq. (14), for $j=4,5$

$$\begin{bmatrix} d_{j1} \\ d_{j2} \end{bmatrix} = \begin{bmatrix} [-3(I_1 - I_2) + (k_2 - 3m) r_1^2]/I_3^M - \omega_j^2 \\ (k_3 - 3m) r_1/(m I_3^M)^{1/2} \end{bmatrix}$$

$$\rho_j^2 (d_{j1}^2 + d_{j2}^2) = 1$$

$\pm\omega_j$ ($j=4,5$) are determined by

$$\begin{vmatrix} [-3(I_1 - I_2) + (k_3 - 3m) r_1^2]/I_3^M - \omega_j^2 & (3m - k_2) r_1/(m I_3^M)^{1/2} \\ (3m - k_2) r_1/(m I_3^M)^{1/2} & k_2/m - \omega_j^2 \end{vmatrix} = 0 \tag{16}$$

Then the linearized equations of motion can be written in the following form

$$\left.\begin{aligned} \dot{q}_j (= \partial H_2/\partial p_j) &= p_j \\ & \qquad (j=1,\dots,5) \\ \dot{p}_j (= -\partial H_2/\partial q_j - \partial F/\partial \dot{q}_j) &\\ = -\omega_j^2 q_j - \partial F/\partial p_j & \end{aligned}\right\} \tag{17}$$

where H_2 is represented by Eq. (12), F is represented by Eq. (5), and p_j and q_j are expressed with the aid of Eq. (3) with $\xi_1 = 0$ and the canonical transformations, Eqs. (10), (13), and (14). For the equilibrium condition $q_j = p_j = 0$ ($j=1,\dots,5$), the damping of every normal mode can be inferred from an examination of the terms $\partial F/\partial p_j$ ($j=1,\dots,5$). Therefore, conditions for pervasive (complete) damping are obtained for the present linear system with semidefinite damping as follows: ($c_2, c_3 > 0$)

$$\partial F(p_k, q_k |_{k=1,\dots,5})/\partial p_j \neq 0 \quad (j=1,\dots,5) \tag{18}$$

or

$$I_3^M + I_1 - I_2 \neq 0 \tag{19a}$$

$$I_1 + I_2 - I_3 - I_3^R r \neq 0 \tag{19b}$$

$$(I_3^R r)^2 - (I_2 - I_3) I_3^R r - 4(I_1 + I_2 - I_3) I_1 \neq 0 \tag{19c}$$

Asymptotic stability conditions for the system with pervasive damping are obtained from the requirement that H_2 be a positive definite function.[8] This is equivalent to requiring the matrices (15) with $\kappa=0$ and (16) with $\omega_j=0$ to be positive, or

$$\Delta_1 \triangleq -I_2+I_3+I_3^R r>0 \tag{20a}$$

$$\Delta_2 \triangleq 4I_3-4I_1+I_3^R r+(k_3-3m)r_1^2>0 \tag{20b}$$

$$\Delta_3 \triangleq \Delta_1[\Delta_2(m+k_3)-(k_3-3m)^2 r_1^2]>0 \tag{20c}$$

$$\Delta_4 \triangleq -3(I_1-I_2)+(k_2-3m)r_1^2>0 \tag{20d}$$

$$\Delta_5 \triangleq \rho_4 k_2-(R_2-3m)^2 r_1^2>0 \tag{20e}$$

Thus the equilibrium state of the present linear system is asymptotically stable if the conditions (19) and (20) are satisfied.

It is noted that the present formulation enables one to examine whether the linear system with semidefinite damping is pervasively damped without recourse to the methods considered by Roberson[9] and Connell.[10]

Parametrically Resonant Librations

It is assumed that the conditions (20) are satisfied and that the generalized coordinates and momenta q_j, p_j $(j=1,\ldots,5)$ are small. If terms up to the second order in these parameters are considered in the Hamiltonian and the damping coefficients c_2 and c_3 are set to zero, the solutions of the equations of motion, Eqs. (17), are obtained in the following form:[11]

$$q_j=(J_j/\pi\omega_j)^{1/2}\sin 2\pi w_j \tag{21a}$$

$$(j=1,\ldots,5)$$

$$p_j=(J_j\omega_j/\pi)^{1/2}\cos 2\pi w_j \tag{21b}$$

where J_j and w_j are given by

$$H_2=\sum_{j=1}^{5}\nu_j J_j \tag{22a}$$

$$J_j=J_j(0) \tag{22b}$$

$$w_j=\nu_j t+\beta_j \triangleq (\omega_j/2\pi)t+\beta_j \tag{22c}$$

and $J_j(0)$, $\nu_j=\omega_j/2\pi$ and β_j are constants.

Taking into account the nonlinear and damping effects, or including the terms up to the third order of q_j and p_j $(j=1,\ldots,5)$ in the Hamiltonian and Rayleigh's dissipation function with $c_2\neq 0$ and $c_3\neq 0$, the equations of variations of the parameters are obtained as follows

$$\dot J_j=-\partial H_3/\partial w_j-Q_j^w \tag{23a}$$

$$(j=1,\ldots,5)$$

$$\dot w_j-\partial H_2/\partial J_j=\partial H_3/\partial J_j+Q_j^J \tag{23b}$$

where

$$\sum_{k=2}^{3}[\partial F(\dot\xi_2,\dot\xi_3)/\partial\dot\xi_r]\delta\xi_k=\sum_{j=1}^{5}(Q_j^J\delta J+Q_j^w\delta w_j)$$

To investigate the nonlinear equations, Eq. (23), the method of averaging is employed, assuming that the values of the terms in the right-hand side of Eq. (23) are small and that they can be expressed by the solutions of the form given in Eqs. (21). Thus H_3, Q_j^J and Q_j^w $(j=1,\ldots,5)$ are averaged over a long period of time. It can be shown that, in the neighborhood of the parametric resonance conditions, i.e., for

$$\nu_4=2\nu_j,\ \nu_5=2\nu_j,\ (j=1,2,3),\ \nu_4=2\nu_5,\ (\nu_4>\nu_5) \tag{24}$$

an interesting phenomenon of motion exists, namely, large nonlinear energy transfer between two modes of motion due to parametric resonance.[2,3] The neighborhood of the condition $\nu_4=2\nu_1$, i.e., $2\nu_1-\nu_4[\triangleq\epsilon]<<1$, is treated below as an example.

The averaged Hamiltonian $\tilde H$ having a long period $1/\epsilon>>1$ can be written as follows:

$$\tilde H=\sum_{j=1}^{5}\tilde\nu_j\tilde J_j+A\tilde J_1\sqrt{\tilde J_4}\cos 2\pi(2\tilde w_1-\tilde w_4) \tag{25}$$

where

$$A\triangleq(\rho_1^2\rho_4/\pi\omega_1\sqrt{\pi\omega_1})\Big[i\{[(I_3+I_3^R r)C_{12}-C_{14}]$$
$$\times C_{15}d_{41}(1/I_1-1/I_2)/\sqrt{I_3^M}$$
$$+k_2r_1C_{11}C_{13}d_{41}/\sqrt{I_3^M}+k_3r_1C_{11}C_{12}d_{42}/\sqrt{m}$$
$$+(k_3-k_2)C_{11}C_{13}d_{42}/\sqrt{m}\}+C_{12}\omega_4\{\sqrt{I_3^M}C_{14}d_{41}/I_1$$
$$-[(I_3+I_3^R r)/I_1-1/2]\sqrt{I_3^M}C_{12}d_{41}-\sqrt{m}(C_{13}+r_1C_{12}/2)d_{42}\}$$
$$-\sqrt{I_3^M}C_{11}^2d_{41}\omega_4\Big]$$

and $(\tilde{\ })$ denotes averaging over a long period of time. The Hamiltonian equations are, for $j=1,\ldots,5$,

$$\dot{\tilde J}_j=-\partial\tilde H_3/\partial\tilde w_j-(c_2D_{2j}+c_3D_{3j})\tilde J_j \tag{26a}$$

$$\dot{\tilde w}_j=\tilde\nu_j+\partial\tilde H_3/\partial\tilde J_j \tag{26b}$$

where

$$D_{2j}=0 \qquad (j=1,2,3)$$

$$D_{2j}=\rho_j^2(r_1d_{j1}/\sqrt{I_3^M}-d_{j2}/\sqrt{m})^2 \qquad (j=4,5)$$

$$D_{3j}=(4\rho_j^2/i\omega_j)(C_{j6}/m+r_1C_{j5}/I_2$$
$$-r_1C_{j1})(r_1C_{j2}+C_{j3}) \qquad (j=1,2,3)$$

$$D_{3j}=0 \qquad (j=4,5)$$

and it assumed that c_2D_{2j} $(j=4,5)$ and c_3D_{3j} $(j=1,2,3)$ are small. Since $\partial\tilde H_3/\partial\tilde J_j=0$ for $j=2,3,5$, we obtain

$$\tilde J_j(t)=\tilde J_j(0)\exp[-(c_2D_{2j}+c_3D_{3j})t] \tag{27a}$$

$$(j=2,3,5)$$

$$\tilde w_j(t)=\tilde\nu_j(0)t+\tilde\beta_j(0) \tag{27b}$$

From Eqs. (27), it is seen that the generalized coordinates and momenta in the non-resonant modes, $\tilde q_j$ and $\tilde p_j$ for $j=2,3,5$, are damped out to zero if the conditions (18), or (19), are satisfied. For the parametrically resonant modes, the analysis is more complicated. Thus, in order to simplify the analysis, the following canonical transformation is employed

$$(\tilde J_1-2\tilde J_4)/4=J,\quad (\tilde J_1+2\tilde J_4)/4=L,$$

$$2\tilde w_1-\tilde w_4=w,\quad 2\tilde w_1+\tilde w_4=v$$

Then the Hamiltonian $\tilde H$ becomes

$$\tilde H=\sum_{j=2,3,5}\tilde\nu_j\tilde J_j+(2\tilde\nu_1-\tilde\nu_4)J+(2\tilde\nu_1+\tilde\nu_4)L+2A$$
$$\times(J+L)(L-J)^{1/2}\cos 2\pi w \tag{29}$$

and the Hamilton equations for the parametrically resonant modes become as follows

$$\dot{J}=4\pi A(J+L)(L-J)^{1/2}\sin 2\pi w-c_2D_{24}(J-L)/2+C_3D_{31}(J+L)/2 \quad (30a)$$

$$\dot{L}=-c_2D_{24}(J-L)/2-c_3D_{31}(J+L)/2 \quad (30b)$$

$$\dot{w}=\partial\tilde{H}/\partial J \quad (30c)$$

$$\dot{v}=\partial\tilde{H}/\partial L \quad (30d)$$

To solve these equations, consider first the conservative system with $c_2=c_3=0$ and

$$H_0\triangleq\tilde{H}-\sum_{j=2,3,5}\tilde{\nu}_j\tilde{J}_j$$

constant. In this case, one of the canonical variables, $L(\triangleq L_0)$, becomes constant, and Eq. (30a) can be written, using Eq. (29), as follows

$$\dot{J}/2\pi=\pm\{4A^2(J+L_0)^2(L_0-J)-[H_0-\epsilon J-(2\tilde{\nu}_1+\tilde{\nu}_4)L_0]^2\}^{1/2} \quad (31)$$

where $\epsilon=2\tilde{\nu}_1-\tilde{\nu}_4<<1$. By setting $\dot{J}=0$ in Eq. (31) and using the fact that H_0 is constant, the extrema of J, or J_m, are determined from

$$H_0=\epsilon J_m+(2\tilde{\nu}_1+\tilde{\nu}_4)L_0+2A(J_m+L_0)(L_0-J_m)^{1/2}\cos N\pi \quad (N=0,\pm 1,\pm 2,\ldots) \quad (32)$$

and Eq. (31) becomes

$$\dot{J}/2\pi=\pm\{4A^2(J+L_0)^2(L_0-J)-[\epsilon(J_m-J)+2A(J_m+L_0)(L_0-J_m)^{1/2}\cos N\pi]^2\}^{1/2} \quad (33)$$

The behavior of solutions of Eq. (33) may be observed in the following manner.† If one of the extreme values, J_{m1} ($>-L_0$) is assumed to be sufficiently close to $-L_0$, or if it is assumed that $\tilde{J}_1|_{J=J_{m1}}=2(J_{m1}+L_0)$ $[>0]\approx 0$ and $\tilde{J}_4|_{J=J_{m1}}=L_0-J_{m1}\approx 2L_0$, another extreme value, J_{m2}, is obtained from Eq. (33) with $\dot{J}=0$ as follows:

$$J_{m2}\approx L_0-\epsilon^2/4A^2 \quad (34)$$

where $J_1|_{J=J_{m2}}\approx 2[2L_0-\epsilon^2/4A^2]$ and $J_4|_{J=J_{m2}}\approx\epsilon^2/4A^2$. Thus, it can be seen that if $2L_0>>\epsilon^2/4A^2$ a large amount of nonlinear energy transfer exists between the modes $\tilde{\nu}_1$ and $\tilde{\nu}_4$ and, on the contrary, if $\epsilon^2/4A^2>2L_0$ no significant nonlinear resonance phenonemon occurs.

When the condition $|\epsilon/A|/2(2L_0)^{1/2}<<1$ is assumed, two types of initial conditions may attract our attention: one is the condition $J_1(0)\approx 0$, or $J(0)\approx -L_0$, where the energy of the modes $\tilde{\nu}_1$ and $\tilde{\nu}_4$ is almost contained in the mode $\tilde{\nu}_4$ (this situation is similar to that treated by Kane[1] and Likins and Wrout.[4]) and another is the condition $J_4(0)\approx 0$, or $J(0)\approx L_0$, where that energy is almost contained in the mode $\tilde{\nu}_1$ (this situation is similar to that treated by Pringle[3]).

Assuming the initial condition $J(0)=J_m=-L_0+\epsilon_1$ $(0<\epsilon_1<<1)$ and that the quantities ϵ_1/L_0 and $|\epsilon/A|/2(2L_0)^{1/2}$ are sufficiently small and their second and higher order terms are negligible, the solution of Eq. (33) can be obtained in terms of elliptic integrals as follows

$$J=L_0-(2L_0+\epsilon_1)\,{}_1k^2sn^2[2A(2L_0)^{1/2}\times(1+\epsilon_1/4L_0)\pi t+K,{}_1k] \quad (35)$$

where $sn(u,{}_1k)$ is the Jacobi elliptic sine, K is the complete elliptic integral of the first kind, ${}_1k^2=1-\epsilon_1/L_0$ and the condition $0<{}_1k^2<1$ or $0<1-\epsilon_1/L_0<1$ is assumed.

Furthermore, assuming the initial condition $J(0)=J_m\approx L_0$ and that $|\epsilon/A|/2(2L_0)^{1/2}$ is sufficiently small that its second and higher order terms are negligible, the solution of Eq. (33) can be obtained as follows

$$J=L_0-[2L_0+|\epsilon/A|(2L_0)^{1/2}/2]\,{}_2k^2\times sn^2[2A[2L_0+|\epsilon/A|(2L_0)^{1/2}/2]^{1/2}\pi t,{}_2k] \quad (36)$$

where

$${}_2k^2=[2L_0-|\epsilon/A|(2L_0)^{1/2}/2]/[2L_0+|\epsilon/A|(2L_0)^{1/2}/2]$$

and the condition $0<{}_2k^2<1$ or $0<|\epsilon/A|/2(2L_0)^{1/2}<1$ is assumed.

Let us now consider the parametrically resonant modes for the non-conservative system with $c_2\neq 0$ and $c_3\neq 0$. The damping behavior of L can be obtained by the formal method of averaging.[12] Assuming that the values of c_2D_{24} and c_3D_{31} are small, the averaged solution $\tilde{L}$ to the first approximation is obtained from the following equation which is constructed from Eq. (30b)

$$\dot{\tilde{L}}=\lim_{T\to\infty}\frac{1}{T}\int_0^T[-c_2D_{24}(J-\tilde{L})/2-c_3D_{31}(J+\tilde{L})/2]\,dt \quad (37)$$

In the above expression, if the initial condition is given by $J(0)=-L(0)+\epsilon_1$, J should be expressed by Eq. (35) with $L_0=\tilde{L}$ and if the initial condition is given by $J(0)\cong L(0)$, J by Eq. (36) with $L_0=\tilde{L}$. Using the relation

$${}_jp\triangleq E/K=\lim_{T\to\infty}\frac{1}{T}\int_0^T[1-{}_jk^2sn^2u]\,dt \quad (j=1,2) \quad (38)$$

where E is the complete elliptic function of the second kind, the solution $\tilde{L}$ of Eq. (37) can be obtained as follows:

1) For the initial condition $J(0)=-L(0)+\epsilon_1$

$$\tilde{L}(t)=\tilde{L}(0)\exp\{-[(1-{}_1\bar{p})c_2D_{24}+{}_1\bar{p}c_3D_{31}]\}+\epsilon_1\cdot{}_1C_n\{\exp\{-[(1-{}_1\bar{p})c_2D_{24}+{}_1\bar{p}c_3D_{31}]\}-1\}/2 \quad (39)$$

2) For the initial condition $J(0)\cong L(0)$

$$(\tilde{L}(t))^{1/2}=(\tilde{L}(0))^{1/2}\exp\{-[(1-{}_2\bar{p})c_2D_{24}+{}_2\bar{p}c_3D_{31}]/2\}+\epsilon\cdot{}_2C_n\left[1-\exp\{-[(1-{}_2\bar{p})c_2D_{24}+{}_2\bar{p}c_3D_{31}]/2\}\right]/(2A)^{1/2} \quad (40)$$

where

$${}_jC_n=(c_2D_{24}-c_3D_{31})/[(1-{}_j\bar{p})c_2D_{24}+{}_j\bar{p}c_3D_{31}] \quad (j=1,2)$$

$${}_j\bar{p}\cong E[{}_jk(\tilde{L}(0))]/K[{}_jk(\tilde{L}(0))]$$

Since in the present situation the value of $\tilde{H}_3$ is small compared with that of $\tilde{H}_2$, the Hamiltonian can be represented by

$$\tilde{H}\cong\tilde{H}_2=\sum_{j=1}^{5}\tilde{\nu}_j\tilde{J}_j \quad (41)$$

†Readers can find in Ref. 3 a method to view Eq. (31) [i.e. to view Eq. (33)] in an instructive manner.

or

$$\tilde{H} \cong \epsilon J + (2\bar{\nu}_1 + \bar{\nu}_4)L + \sum_{j=2,3,5} \bar{\nu}_j \tilde{J}_j \tag{42}$$

Assuming that the values of the terms containing ϵ_1 and ϵ in Eqs. (39), (40), and (42) are sufficiently small, the Hamiltonian is approximately written by

$$\tilde{H} \cong (2\bar{\nu}_1 + \bar{\nu}_4)\tilde{L} + \sum_{j=2,3,5} \bar{\nu}_j \tilde{J}_j \tag{43}$$

and, using Eqs. (27), (39), and (40), we can finally obtain an expression for the Hamiltonian as follows

$$\tilde{H} = (2\bar{\nu}_1 + \bar{\nu}_4)[\tilde{J}_1(0)/4 + \tilde{J}_4(0)/2]\exp\{-[(1-{}_j\bar{p})c_2D_{24} + {}_j\bar{p}c_3D_{31}]t\} + \sum_{k=2,3,5} \bar{\nu}_k \tilde{J}_k(0)\exp[-(c_2D_{2k} + c_3D_{3k})t] \tag{44}$$

both for the initial conditions $\tilde{J}_1(0) = 2[J(0) + L(0)] = 2\epsilon_1 << 1$ $({}_j\bar{p} = {}_1\bar{p})$ and $\tilde{J}_4(0) = L(0) - J(0) \cong 0$ $({}_j\bar{p} = {}_2\bar{p})$.

The damping behavior of the satellite librations can approximately be observed from the decrease of Hamiltonian of the system, since H_2, which dominates in the value of H, is a positive-definite function of these deviations and their time derivatives [see Eqs. (11)-(14)]. Therefore, Eq. (44) shows the attenuation of the librations when parametric resonance occurs.

This results may be compared with the case when no parametric resonance exists and the Hamiltonian of the system is expressed to the second order as follows

$$\tilde{H} = \bar{\nu}_1\tilde{J}_1(0)\exp(-c_3D_{31}t) + \bar{\nu}_4\tilde{J}_4(0)\exp(-c_2D_{24}t) + \sum_{j=2,3,5} \bar{\nu}_j\tilde{J}_j(0)\exp[-(c_2D_{2j} + c_3D_{3j})t] \tag{45}$$

Comparing Eqs. (44) and (45), it can be seen that: 1) if $\tilde{J}_4(0)/\tilde{J}_1(0) << 1$, the value of $\tilde{J}_1$ in the parametrically resonant case decreases faster (slower) than that in the nonresonant case when $c_2D_{24} > c_3D_{31}$ $(c_3D_{31} > c_2D_{24})$; 2) if $\tilde{J}_1(0)/\tilde{J}_4(0) << 1$, the value of $\tilde{J}_4$ in the parametrically resonant case decreases faster (slower) than that in the nonresonant case when $c_3D_{31} > c_2D_{24}$ $(c_2D_{24} > c_3D_{31})$; 3) when $c_2 \cdot D_{24} = c_3D_{31}$, for both of the initial conditions $\tilde{J}_4(0)/\tilde{J}_1(0) << 1$, and $\tilde{J}_1(0)/\tilde{J}_4(0) << 1$, the values of $\tilde{J}_1$ and $\tilde{J}_4$ decrease in the same speed both in the parametrically resonant and the nonresonant cases.

This observation shows that for parametric resonance between two modes, the action variable $J_1(0)$ $[J_4(0)]$ is damped out due to the significant influence of energy dissipation in another mode corresponding to c_2D_{24} $[c_3D_{31}]$.

Comparison with Numerical Solution

The approximate solution (44) obtained analytically is now compared with Eq. (25) obtained by solving numerically Eqs. (26) with the initial conditions $\tilde{J}_2(0) = \tilde{J}_3(0) = \tilde{J}_5(0) = 0$. To simplify the numerical integration of Eqs. (26), the following canonical transformation is introduced [3]

$$x_1 = (\tilde{J}_1/2\pi)^{1/2}\sin 4\pi\tilde{w}_1, \quad y_1 = (\tilde{J}_1/2\pi)^{1/2}\cos 4\pi\tilde{w}_1 \tag{46a}$$

$$x_4 = (\tilde{J}_4/\pi)^{1/2}\sin 2\pi\tilde{w}_4, \quad y_4 = (\tilde{J}_4/\pi)^{1/2}\sin 2\pi\tilde{w}_4 \tag{46b}$$

and the following equations of variations of parameters x_1, y_1, x_4 and y_4 are employed in place of Eqs. (26)

$$\dot{x}_1 = \partial\tilde{H}_3/\partial y_1 - (c_3D_{31}/2)x_1 \tag{47a}$$

$$\dot{y}_1 = -\partial\tilde{H}_3/\partial x_1 - (c_3D_{31}/2)y_1 \tag{47b}$$

$$\dot{x}_4 = \partial\tilde{H}_3/\partial y_4 - (c_2D_{24}/2)x_4 \tag{47c}$$

$$\dot{y}_4 = -\partial\tilde{H}_3/\partial x_4 - (c_2D_{24}/2)y_4 \tag{47d}$$

where

$$\tilde{H}_3 = 2\pi A[\pi(x_1^2 + y_1^2)]^{1/2}(y_1y_4 + x_1x_4) \tag{48}$$

As an example, the configuration parameters of the system are chosen as follows

$$(I_1 - I_2)/I_3 = -0.9,\ (I_2 - I_3)/I_1 = -0.5,\ I_3^R/I_3 = 0.1,$$

$$I_3^R r/I_3 = 0.173,\ [mr_1^2/I_3]^{1/2} = 0.00076,\ k_3/m = 0,$$

$$[k_2/m]^{1/2} = 0.00001,\ A \cong 0.171$$

where $\epsilon = 2\nu_1 - \nu_4 = 0.00152$. In Figs. 2a-5a, the approximate solution (44) obtained in the present analysis is compared with the numerical solution obtained from Eqs. (47). It should be noted that in Figs. 2-5 the values of $\tilde{H}$, $\tilde{J}_j$, J and L are shown with their values normalized by I_3.

Figures 2 and 3 show the long period motion of the type $\tilde{J}_1(0)/\tilde{J}_4(0) << 1$ with damping $c_3D_{31} = 0.033$ and 0.082, respectively, and with the initial conditions $\tilde{J}_1(0) = 0.01$ and $\tilde{J}_4(0) = 0.25$ $(\epsilon_1 = 0.005)$. The value of c_2 is selected to be zero in order to clarify the damping due to parametric resonance. The behavior of the Hamiltonian $\tilde{H}$ is shown in Figs. 2a and 3a and, using numerical integration of Eqs. (47), the behavior of the action variables $\tilde{J}_1$, $\tilde{J}_4$, J and L are shown in Figs. 2b and 3b. It is seen that the variation of L is rather moderate

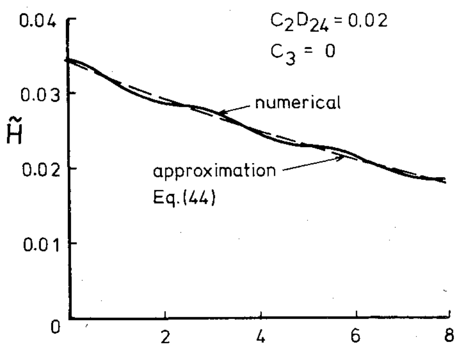

Fig. 4a Comparison between analytically and numerically predicted behaviors of $\tilde{H}$: plots against time in orbits.

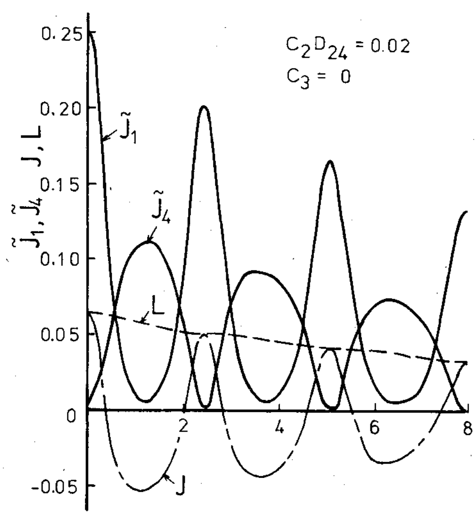

Fig. 4b Plots of $\tilde{J}_1$, $\tilde{J}_4$, J and L against time in orbits.

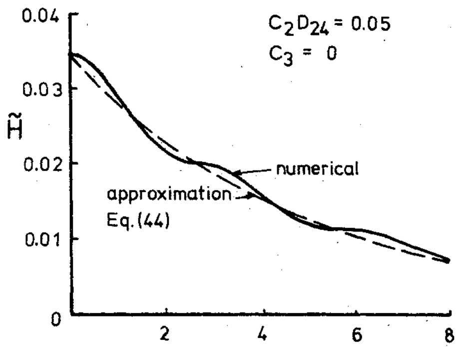

Fig. 5a Comparison between analytically and numerically predicted behaviors of $\tilde{H}$: plots against time in orbits.

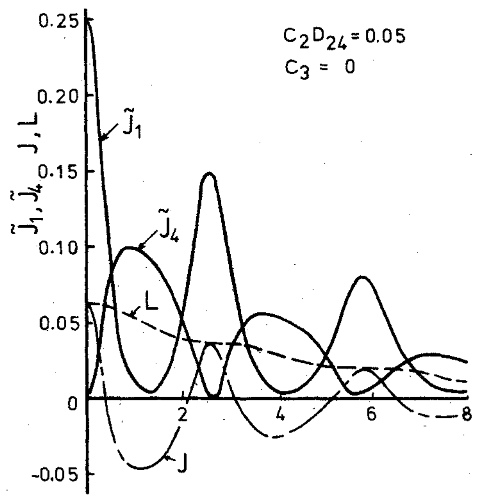

Fig. 5b Plots of $\tilde{J}_1$, $\tilde{J}_4$, J and L against time in orbits.

with respect to time in comparison with those of J_1, J_4 and J and it is similar to that of H.

Figures 4 and 5 show the long period motion of the type $J_4(0)/J_1(0) << 1$ with damping $c_2D_{24} = 0.02$ and 0.05, respectively. The initial conditions are $J_1(0) = 0.25$ and $J_4(0) = 0.0001$ and it is assumed that $c_3 = 0$.

From Figs. 2a-5a, it is shown that by the approximate solution (44) it is possible to predict the decrease in the Hamiltonian of the damped librations of the parametrically resonant satellite.

Conclusion and Recommendations

In the present paper, a mathematical model is formulated to analyze the attitude dynamics of a dual-spin satellite with a gravity-oriented main body and a damper mass. It is shown that the canonical transformation method can be easily applied to the present system by describing damper motions in terms of coordinates parallel to the orbital frame and then introducing constraint equations.

The effect of parametric resonance on the attenuation of satellite librations has been investigated. It is shown that the rapidly time-varying canonical variables (e.g. J_1 and J_4) of two parametrically resonant modes can be transformed into the moderately time-varying (L) and rapidly time-varying (J) canonical variables; the variation of the Hamiltonian can be obtained analytically by making use of the resulting moderately time-varying canonical variable. The effect of parametric resonance on the damped motion is observed via the decrease of the Hamiltonian of the system.

The attenuation of librations has been analyzed and its features in the case of parametric resonance and of no parametric resonance are compared. The configuration parameters of the satellite which lead to faster damping of these two cases are also investigated.

An analysis of the linearized system shows that the present mathematical model provides a useful technique of obtaining the conditions for pervasive damping of attitude motion for the class of satellites treated here.

It may be seen that the present procedure of analysis is applicable to attitude dynamics studies on a wide class of two-body satellites including both linear and nonlinear motions and that the extension to many-body satellites can also be expected.

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